Lecture 12: Graph Search
The intellectual themes in 6.01 are recurring themes in engineering:

- design of complex systems
- modeling and controlling physical systems
- augmenting physical systems with computation
- building systems that are robust to uncertainty
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Approach: focus on key concepts to pursue in depth
6.01: Introduction to EECS I

The **intellectual themes** in 6.01 are recurring themes in engineering:

- design of complex systems
- modeling and controlling physical systems
- augmenting physical systems with computation
- building systems that are robust to uncertainty

**Approach:** focus on **key concepts** to pursue **in depth**
Module 1: Signals and Systems

Modeling and analyzing behavior of physical systems

**Topics:** Feedback Control Systems

**Lab Exercises:** Wall-finder, Wall-follower, Jousting
Module 2: Circuits

Designing, constructing, and analyzing physical systems

Topics: Resistive Networks, Op-Amps, Linearity and Equivalence

Lab Exercises: Design a new sensory modality for the robot
Module 3: Bayesian Reasoning

Modeling uncertainty and designing robust systems

**Topics:** Subjective Probability, Bayesian Inference

**Lab Exercises:** Localization and Parking
Module 4: Planning

Augmenting physical systems with computation.

Topics: Graph Search

Lab Exercises: Solving mazes, Path planning on maps
Module 4: Planning

Augmenting physical systems with computation.

**Topics:** Graph Search

**Lab Exercises:** Solving mazes, Path planning on maps
Graph Search (Path Planning)

What is a graph?

- Some set $V$ of vertices
- Some collection $E$ of edges connecting vertices
Example: 8-Puzzle

Start

1 2 3
4 5 6
7 8

Goal

1 2
3 4 5
6 7 8
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle

```
1 2 3
|   |
4 5 6
|   |
7 8 |
```
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle

![8-Puzzle Diagram](image-url)
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle

```
1  4  2
7  6  3
8  5
```
Example: 8-Puzzle
Example: 8-Puzzle

![8-Puzzle Diagram](image-url)
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle

```
1 4 2
6 3 5
7 8
```
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Example: 8-Puzzle
Check Yourself

How many different board configurations (states) exist?

1. \(8^2 = 64\)
2. \(9^2 = 81\)
3. \(8! = 40,320\)
4. \(9! = 362,880\)
5. None of the above
Check Yourself

How many different board configurations (states) exist?

1. \(8^2 = 64\)
2. \(9^2 = 81\)
3. \(8! = 40,320\)
4. \(9! = 362,880\)
5. None of the above
Check Yourself

We could have to look through as many as \(9! = 362,880\) states (or even more if we’re not careful!)

- And we would have to figure out for each state, which of the other states can be reached through a single move.

Is the solution with 22 moves optimal? Do shorter solutions exist?
Do we have to look at all 362,880 configurations to be sure?
Graph Search

In this module:

- Develop algorithms to systematically “search” through a graph
- Analyze how well the algorithms perform
Graph Search

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- Analyze how well the algorithms perform
- Optimize the algorithms:
  - Find “better” paths (results)
  - Consider fewer cases (speed)
Graph Search

In this module:

- Develop algorithms to systematically “search” through a graph

- Analyze how well the algorithms perform

- Optimize the algorithms:
  - Find “better” paths (results)
  - Consider fewer cases (speed)

- Observe the algorithms at work in multiple contexts
  - Robot path-planning
  - Route Planning in USA
  - Language
  - Biology
Example: Grid Search

Find path between 2 points on a rectangular grid.

A — B — C

D — E — F

G — H — I
Example: Grid Search

Find path between 2 points on a rectangular grid.

Represent all possible paths from A with a tree:
Notice that there are infinitely many paths.
The tree is infinitely large!
Notice that there are infinitely many paths.

The tree is infinitely large!

Strategy: construct the tree **incrementally** while looking for a path
Represent the grid as instance of class `Grid`

```python
class Grid:
    def __init__(self, width, height, start, goal):
        self.width = width
        self.height = height
        self.start = start
        self.goal = goal

grid = Grid(3, 3, (0,0), (2,2))
```
Search Trees in Python

Represent each node in the tree as an instance of SearchNode

Note: no explicit representation for entire tree

Issues:
- need to “grow” the tree as we search it
- need to reconstruct paths in tree
Represent each **node** in the tree as an instance of `SearchNode`

```python
class SearchNode:
    def __init__(self, state, parent):
        self.state = state
        self.parent = parent

    def path(self):
        p = []
        node = self
        while node:
            p.append(node.state)
            node = node.parent
        return p[::-1]
```
Pathfinding Algorithm

Construct the tree and find a path to the goal.

Algorithm:
- Initialize *agenda* (list of nodes to consider)
- Repeat the following:
  - Remove one node from the agenda ("expand")
  - Add that node’s successors to the agenda ("visit")
until *goal is found* or *agenda is empty*
- Return resulting path
Order Matters!

Strategy: Replace last node in agenda by its successors
Order Matters!

Strategy: Replace last node in agenda by its successors

Agenda: A
Order Matters!

Strategy: Replace last node in agenda by its successors

Agenda: A AB AD
Order Matters!

Strategy: Replace last node in agenda by its successors

Agenda: A AB ABD ADA ADE ADG
Order Matters!

Strategy: Replace last node in agenda by its successors

Agenda: A AB AD AΔ ADA ADE AΔG ADGD ADGH
Order Matters!

Strategy: Replace last node in agenda by its successors

Agenda: A AB AÐ ADA ADE AÐG ADGD ADGH

Depth-first Search
Order Matters!

Strategy: Replace first node in agenda by its successors
Order Matters!

Strategy: Replace first node in agenda by its successors

Agenda: A
Order Matters!

Strategy: Replace first node in agenda by its successors

Agenda: A AB AD
Order Matters!

Strategy: Replace first node in agenda by its successors

Agenda: A AB ABA ABC ABE AD
Order Matters!

Strategy: Replace first node in agenda by its successors

Agenda: A AB ABA ABAB ABAD ABC ABE AD
Order Matters!

Strategy: Replace first node in agenda by its successors

Agenda: A AB ABA ABAB ABAD ABC ABE AD

Still Depth-first Search
Order Matters!

Strategy: Remove first node and add its successors to end
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD ABA ABC ABE
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD ABA ABC ABE ADA ADE ADG
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD ABA ABC ABE ADA ADE ADG ABAB ABAD
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD ABA ABC ABE ADA ADE ADG ABAB ABAD ABCB ABCF
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD ABA ABC ABE ADA ADE ADG ABAB ABAD ABCB ABCF ABEB ABED ABEF ABEH
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD AÐ ABA ABC ABE ADA ADE ADG ABAB ABAD ABCB ABCF ABEB ABED ABEF ABEH ADAB ADAD
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD AÐ ABA ABC ABE AÐA AÐE ADG ABAB ABAD ABCB ABCF ABEB ABED ABEF ABEH ADAB ADAD ADEB ADED ADEF ADEH
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD AØ ABA ABC ABE AØA AØE AØG ABAB ABAD ABCB ABCF ABEB ABED ABEF ABEH ADAB ADAD ADEB ADED ADEF ADEH ADGD ADGH
Order Matters!

Strategy: Remove first node and add its successors to end

Agenda: A AB AD ABA ABC ABE ADA ADE ADG ABAB ABAD ABCB ABCF ABEB ABED ABEF ABEH ADAB ADAD ADEB ADED ADEF ADEH ADGD ADGH

*Breadth-first* Search
Order Matters!

Depth-First Search (DFS):
- Push and Pop from same side of agenda
- Works down one branch of the tree before moving on to another branch

Breadth-First Search (BFS):
- Push and Pop from different sides of agenda
- Considers all paths of length $n$ before considering paths of length $n+1$
Too Much Searching

Find path between 2 points on a rectangular grid.

Represent all possible paths with a tree:

But don’t need to consider all nodes!
Pruning

“Prune” the tree to reduce the amount of work.

**Pruning Strategy 1:**
Don’t consider any path the visits the same state twice.

Algorithm:
- Initialize agenda (list of nodes to consider)
- Repeat the following:
  - Remove one node from the agenda
  - Add each child (of that node) to the agenda if its state is not in the parent’s path
- until goal is found or agenda is empty
- Return resulting path
Pruning

“Prune” the tree to reduce the amount of work.

**Pruning Strategy 1:**
Don’t consider any path the visits the same state twice.

Algorithm:
- Initialize *agenda* (list of nodes to consider)
- Repeat the following:
  - Remove one node from the agenda
  - Add each child (of that node) to the agenda *if its state is not in the parent’s path.*
- Until *goal is found* or *agenda is empty*
- Return resulting path
Pruning

“Prune” the tree to reduce the amount of work.

**Pruning Strategy 1:**
Don’t consider any path that contains the same state twice.
“Prune” the tree to reduce the amount of work.

**Pruning Strategy 1:**
Don’t consider any path that contains the same state twice.
Pruning

“Prune” the tree to reduce the amount of work.

**Pruning Strategy 1:**
Don’t consider any path that contains the same state twice.
Pruning

Under strategy 1, BFS in the 3x3 grid still visits 16 nodes...
Pruning

Under strategy 1, BFS in the 3x3 grid still visits 16 nodes... but there are only 9 states!

We should be able to reduce the search even further.
Dynamic Programming

Basic idea behind dynamic programming:

- Break big problem into easy ones, solve and combine.
- Remember the solutions to the easy problems for later use.
Dynamic Programming

Basic idea behind dynamic programming:

- Break big problem into easy ones, solve and combine.
- Remember the solutions to the easy problems for later use.

Appropriate if problem has:

- **optimal substructure**: best solution is combination of optimal solutions to sub-problems
- **overlapping sub-problems**: same sub-problem occurs many times while solving overall problem
Dynamic Programming

As applies to search:
(Depends slightly on which algorithm we’re using)
Dynamic Programming

As applies to search:
(Depends slightly on which algorithm we’re using)

**BFS:** The shortest path $S \to X \to G$ is made up of the shortest path $S \to X$ and the shortest path $X \to G$. 
Dynamic Programming

As applies to search:
(Depends slightly on which algorithm we’re using)

**BFS:** The shortest path $S \rightarrow X \rightarrow G$ is made up of the shortest path $S \rightarrow X$ and the shortest path $X \rightarrow G$.

**DFS:** A path $S \rightarrow X \rightarrow G$ is made up of a path $S \rightarrow X$ and a path $X \rightarrow G$. 
Dynamic Programming

As applies to search:
(_depends slightly on which algorithm we’re using)

**BFS:** The shortest path $S \rightarrow X \rightarrow G$ is made up of the shortest path $S \rightarrow X$ and the shortest path $X \rightarrow G$.

**DFS:** A path $S \rightarrow X \rightarrow G$ is made up of a path $S \rightarrow X$ and a path $X \rightarrow G$.

The moral: once we have found a path $S \rightarrow X$, we don’t need to spend time looking for other paths through $X$. 
Dynamic Programming

As applies to search:
(Depends slightly on which algorithm we’re using)

**BFS:** The shortest path $S \rightarrow X \rightarrow G$ is made up of the shortest path $S \rightarrow X$ and the shortest path $X \rightarrow G$.

**DFS:** A path $S \rightarrow X \rightarrow G$ is made up of a path $S \rightarrow X$ and a path $X \rightarrow G$.

The moral: once we have found a path $S \rightarrow X$, we don’t need to spend time looking for other paths through $X$.

Said another way: Many paths that include $S \rightarrow X$, but don’t need to recompute while exploring rest of path (*memoization*); and once have a satisfactory path $S \rightarrow X$, don’t need to keep looking for others (*dynamic programming*).
Dynamic Programming

As applied to graph search: Don’t consider any path that visits a state that you have already visited via some other path.

Need to remember which states we have visited to avoid visiting them again.

Algorithm:
• Initialize visited set
• Initialize agenda (list of nodes to consider)
• Repeat the following:
  - Remove one node from the agenda
  - Add each child (of that node) to the agenda if its state is not already in the visited set, and add each of these new states to the visited set
• Until goal is found or agenda is empty
• Return resulting path
Dynamic Programming

As applied to graph search: Don’t consider any path that visits a state that you have already visited via some other path.

Need to remember which states we have visited to avoid visiting them again.

Algorithm:

- Initialize \textit{visited set}
- Initialize \textit{agenda} (list of nodes to consider)
- Repeat the following:
  - Remove one node from the agenda
  - Add each child (of that node) to the agenda \textit{if its state is not already in the visited set}, and \textit{add each of these new states to the visited set}
  until \textit{goal is found} or \textit{agenda is empty}
- Return resulting path
Check Yourself!

Consider a breadth-first search with dynamic programming, from A to I. How many states are visited?

1. 2
2. 4
3. 6
4. 8
5. 10
Consider a breadth-first search with dynamic programming, from $A$ to $I$. How many states are visited?

1. 2
2. 4
3. 6
4. 8
5. 10
Before we continue...

Properties of DFS/BFS
Grid: BFS 1
Grid: BFS 1
Grid: BFS 1
Grid: BFS 1
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Grid: BFS 1
Grid: BFS 1
Grid: BFS 1
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Grid: DFS 2

[Diagram of DFS traversal on a grid]

29 Apr 2019
Grid: DFS 2
Grid: DFS 2
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Grid: DFS 2
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Grid: DFS 2
Grid: DFS 2
Grid: DFS 2
Grid: DFS 2
Grid: DFS 2
Grid: DFS 2
quiz
quit
suit
suet
sues
sups
sops
soys
Words: DFS

says
words
sawn
sown
soon
soot
sort
sore
some
same
save
wave
wavy
waxy
wary
wart
watt
want
wand
wind
wins
wits
with
wish
wisp
wasp
rasp
rash
rush
rust
runt
runs
ruts
rots
rote
rove
rive
rise
risk
rink
ring
rang
sang
sank
sack
such
ouch
much
mach
mace
maze
raze
razz
jazz
rare
tare
tars
vats
vans
vane
vast
vest
best
quiz
quid
quip
quit
quad
suit
quay
skit
slit
spit
suet
skid
skim
skin
skip
Words: BFS

alit
flit
slot
slid
slim
slip
knit
unit
Words: BFS

snip
spat
spot
stet
sued
sues
said
shim
swim
akın
shin
ship
alia
flat
flip
plat
scat
seat
slab
slag
slam
slap
slav
slaw
slay
blot
clot
plot
scot
shot
soot
sloe
slog
slop
slow
sled
slum
clip
knot
snap
span
spar
Words: BFS

spas
spay
spun
diet
duct
dust
duel
dues
stem
step
stew
cued
hued
rued
seed
shed
sped
surd
hues
Words: BFS

rues
sees
subs
sums
suns
sups
laid
maiden
paid
sand
sail
whim
sham
swam
swum
Words: BFS

chin
thin
Words: BFS

shun
chip
whip
shop
ilia
aria
asia
alga
Words: BFS

alma
feat
fiat
Words: BFS

flab
flag
Words: BFS

flak
flap
flaw
flax
flay
flop
peat
plan
Words: BFS

play
scab
scan
scar
Words: BFS

beat
best
Words: BFS vs DFS

BFS
quiz → quit → suit → slit → slat → seat → beat → best

DFS
quiz → quit → suit → suet → sue → sups → sops → soys → says → saws →
sawn → sown → soon → soot → sort → sore → some → same → save →
wave → wavy → wary → wart → want → wand → wind → wins → wits →
with → wish → wisp → wasp → rasp → rash → rush → rust → runt → runs →
ruts → rots → rote → rove → rive → rise → risk → rink → ring → rang →
sang → sank → sack → suck → such → much → mach → mace → maze →
raze → rare → tare → tars → tats → vats → vans → vane → vase → vast →
vest → best
lib601 procedure called search, takes arguments:

- **successors**: function that takes a state and returns a list of successor states
- **start_state**: the state from which to start the search
- **goal_test**: a function that takes a state and returns True if that state satisfies the goal condition, and False otherwise
- **dfs**: boolean; if True, run a depth-first search; if False, run a breadth-first search

search returns a list of states from the root of the tree to the goal, or None if no path exists.
def search(successors, start_state, goal_test, dfs = False):
    if goal_test(start_state):
        return [start_state]
    else:
        agenda = [SearchNode(start_state, None)]
        visited = {start_state}
        while len(agenda) > 0:
            parent = agenda.pop(-1 if dfs else 0)
            for child_state in successors(parent.state):
                child = SearchNode(child_state, parent)
                if goalTest(child_state):
                    return child.path()
                if child_state not in visited:
                    agenda.append(child)
                    visited.add(child_state)
        return None
def search(successors, start_state, goal_test, dfs = False):
    if goal_test(start_state):
        return [start_state]
    else:
        agenda = [SearchNode(start_state, None)]
        visited = {start_state}
        while len(agenda) > 0:
            parent = agenda.pop(-1 if dfs else 0)
            for child_state in successors(parent.state):
                child = SearchNode(child_state, parent)
                if goalTest(child_state):
                    return child.path()
                if child_state not in visited:
                    agenda.append(child)
                    visited.add(child_state)
        return None
def search(successors, start_state, goal_test, dfs = False):
    if goal_test(start_state):
        return [start_state]
    else:
        agenda = [SearchNode(start_state, None)]
        visited = {start_state}
        while len(agenda) > 0:
            parent = agenda.pop(-1 if dfs else 0)
            for child_state in successors(parent.state):
                child = SearchNode(child_state, parent)
                if goalTest(child_state):
                    return child.path()
                if child_state not in visited:
                    agenda.append(child)
                    visited.add(child_state)
        return None
Casting Problems as Search Problems

Biggest issue is choice of state.
Casting Problems as Search Problems

Biggest issue is choice of state.

From the state, we must be able to:

- Determine successors
- Test for goal condition

We’ll get a lot of practice with this during the labs and homeworks this week.
Example: Grid Search

```
class Grid:
    def __init__(self, width, height, start, goal):
        self.width = width
        self.height = height
        self.start = start
        self.goal = goal

grid = Grid(3, 3, (0,0), (2,2))

def grid_successors(state):
    r,c = state
    out = []
    for (dr,dc) in [(0,1),(1,0),(0,-1),(-1,0)]:
        if 0<=(r+dr)<grid.height and 0<=(c+dc)<grid.width:
            out.append((r+dr,c+dc))
    return out

result = search(grid_successors, grid.start, lambda x: x==grid.goal, False)
```
Recap

Developed two search algorithms:
- Breadth-first search
- Depth-first search

Discussed the benefits and drawbacks of each

Developed two pruning rules:
- Don’t consider paths that revisit states
- Only consider the first path to a given state
Labs This Week

**Software Lab:** Solving Mazes  
**Design Lab:** Robots in Mazes